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# Aspects of affine Toda field theory

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Several of the recently discovered classical and quantum features of affine Toda field theory are briefly reviewed, with particular emphasis on the Lie algebraic structure of masses, conserved quantities and $S$-matrices.

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## 1. The model

Affine Toda field theory [1,2] is a theory of $r$ scalar fields in two-dimensional Minkowski space–time, where $r$ is the rank of a compact semi-simple Lie algebra $g$. The classical field theory is determined by the lagrangian density

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a-V(\phi)\ , \qquad (1.1)$$

$$V(\phi)=\frac{m^2}{\beta^2}\sum_0^r n_i \mathrm{e}^{\beta\alpha_i\cdot\phi}\ . \qquad (1.2)$$

In (1.2), $m$ and $\beta$ are real, classically unimportant constants, $\alpha_i$, $i=1, ..., r$, are the simple roots of the Lie algebra $g$, and $\alpha_0=\sum_1^r n_i\alpha_i$ is a linear combination of the simple roots; it corresponds to the extra spot on an extended Dynkin diagram for $g$. If the term $i=0$ is omitted from (1.2) in the lagrangian (1.1), then the theory, both classically and after quantisation, is conformal; with the term $i=0$, the conformal symmetry is broken but the theory remains classically integrable, in the sense that there are infinitely many independent conserved charges in involution.

A reasonable question to ask is whether the classical integrability survives into the quantum field theory and, if so, what is the spectrum and to what extent is it possible to calculate explicitly quantities of interest such as $S$-matrices and form factors. Beginning with the article of Arinshtein, Fateev and Zamolodchikov of some years ago [1], and stimulated recently by questions concerning integrable perturbations of conformal field theory [3,4], a number of very interesting facts have been discovered concerning these special two-dimensional field theories [5–14]. The recent discoveries leave no doubt that these relatively simple models have much structure and their study (even in the $\beta^2>0$ regime) will be informative.

In this short review, the ADE series of Lie algebras will be singled out for special attention. More is known about these than the other "non simply laced" cases although in the end the ADE series may be found to be rather a special set of cases in a wider structure. Work on the non simply laced cases is beginning to be exciting [15,16].

## 2. Classical facts

The classical integrability of the affine Toda field theories relies on the existence of a Lax pair from which the conserved quantities may be established. The details of this are a story in itself [2] but from our present perspective it is enough to be aware of one of the main results. Namely, the conserved charges are two dimensional Lorentz-tensors, labelled by their "spin" in light-cone coordinates, the possible spins being the exponents of the algebra repeated modulo its Coxeter number $h=\sum_0^r n_i$. In other words, the conserved charges may be denoted $Q_{s+kh}$, where $s$ is an exponent and $k$ is an integer. The quantities $Q_{\pm 1}$ correspond to the light-cone components of the energy–momentum. If the quantised field theory retains the integrability property, it is expected that the conserved quantities will survive as mutually commuting quantum operators whose eigenstates are the particles of the theory. Thus, for single-particle states,

$$Q_p\,|\mathrm{a}\rangle = q_p^{\mathrm{a}} e^{p\theta_{\mathrm{a}}}|\mathrm{a}\rangle\,, \quad p=s+kh\,, \tag{2.1}$$

where $\theta_{\mathrm{a}}$ is the rapidity of the particle labelled a,

$$p_{\mathrm{a}} \equiv m_{\mathrm{a}}\,(\cosh\theta_{\mathrm{a}}, \sinh\theta_{\mathrm{a}})\,, \tag{2.2}$$

and $m_{\mathrm{a}}$ is its mass.

Taking the classical lagrangian as the starting point for the definition of a quantum field theory, the classical masses can be computed on expanding the potential (1.2) as far as the quadratic term. Thus the mass matrix is

$$(M^2)^{\mathrm{ab}} = m^2 \sum_0^r n_i \alpha_i^{\mathrm{a}} \alpha_i^{\mathrm{b}}\,. \tag{2.3}$$

For most cases, the mass matrix was diagonalised some time ago [2]. However, more recently, it was noticed [6,13] and then proved Lie algebraically [17], that the eigenvalues of the mass matrix $m_{\mathrm{a}}^2$ were themselves the squares of the components of the lowest-eigenvalue eigenvector of the Cartan matrix corresponding to $g$. In other words, it is possible to choose an ordering of the masses so that $\boldsymbol{m}=(m_1, m_2, ..., m_r)$ and

$$Cm=4\sin^2(\pi/2h)\,m\,. \tag{2.4}$$

This is quite a remarkable result since it allows the particles to be assigned unambiguously (up to mass degeneracies), to the Dynkin diagram for $g$. Even more remarkably, for the ADE series of simply laced algebras (and for one of the twisted cases $a^{(2)}_{\mathrm{even}}$), the classical mass ratios are preserved in perturbative field theory at least to one-loop order [7,11], suggesting in turn that the set of eigenvalues $q^{\mathrm{a}}_1$ in (2.1) is an eigenvector of the Cartan matrix for $g$. In a while, a generalisation of this result will be discussed.

Again at the classical level, it is interesting to examine the cubic term in the expansion of (1.2) since this defines the classical three-point couplings, needed to carry out, for example, the one-loop check mentioned above. Once the mass eigenstates are known, it is possible to compute the couplings, $c^{abc}=\sum_0^r n_i\alpha_i^{\mathrm{a}}\alpha_i^{\mathrm{b}}\alpha_i^{\mathrm{c}}$. For many triples, the coupling vanishes. However, when the coupling is not zero it is proportional always [7,11] to the area of a triangle whose sides have lengths equal to the masses of the three participating particles a, b, c. One consequence of this is that the coupling defines a set of angles (the angles in the triangle) by, for example,

$$m_{\mathrm{c}}^2=m_{\mathrm{b}}^2+m_{\mathrm{b}}^2-2m_{\mathrm{a}}m_{\mathrm{b}}\cos\bar{\theta}^{\mathrm{c}}_{\mathrm{ab}}\,, \tag{2.5}$$

where

$$\bar{\theta}=\pi-\theta\,. \tag{2.6}$$

(There is a convention in the literature that the outside angles of the triangle are denoted by $\theta^{\mathrm{c}}_{\mathrm{ab}}$, etc.) Just which couplings are non-zero will be explained further below.

## 3. Quantum conjectures

It has already been mentioned that the real questions of interest do not concern the classical theory at all. The quantum theory will be regarded as integrable if the conserved quantities $Q_p$ survive and label the particle states. However, it is not easy to establish the quantum integrability using perturbative techniques. Rather these are to be supplemented by several conjectures which are used to compute interesting quantities, such as scattering matrices, whose features may be checked to low orders of perturbation theory. The essential features of the ideas used have been summarised some time ago in a review by Zamolodchikov and Zamolodchikov [18].

Besides supposing the particle states to be eigenstates of the conserved quantities, one of the main ideas [19] concerns the fusing of two particles to form a "bound state". In other words, assuming the particle states may be defined for complex rapidities, there may be points in the variable $\theta_{ab}=\theta_a-\theta_b$, the relative rapidity of a pair of particles a, b, at which the two-particle state $|a, b\rangle$ shares the quantum numbers of the single-particle state $|\bar{c}\rangle$. (Although in the initial lagrangian the fields were all real, the mass eigenstates are not necessarily real in those circumstances where there are mass degeneracies, e.g., in the A series of theories; in those cases the bar denotes the conjugate particle.) In other words,

$$|a, b\rangle \approx |\bar{c}\rangle \,, \tag{3.1}$$

when $\theta_{ab}=\mathrm{i}U^c_{ab}$. The latter will be referred to as "fusing" angles. The other two fusings ac→$\bar{b}$ and bc→$\bar{a}$ are also possible, their fusing angles satisfying

$$U^c_{ab}+U^b_{ac}+U^a_{bc}=2\pi \,.$$

Using (2.1) in conjunction with (3.1) leads to relations between the eigenvalues of the conserved quantities and the fusing angles. Thus, for each possible fusing

$$q^a_p e^{-ip\bar{U}^b_{ac}}+q^b_p e^{ip\bar{U}^a_{bc}}=q^{\bar{c}}_p \,. \tag{3.2}$$

On general grounds $q^{\bar{c}}_p=(-)^{p+1}q^c_p$ (also implying that self-conjugate particles have zero eigenvalue for charges of even spin), so that (3.2) may be rearranged to

$$q^b_p+q^c_p e^{ipU^a_{bc}}+q^a_p e^{ip(U^a_{bc}+U^b_{ac})}=0 \,, \tag{3.3}$$

which represents a set of triangles, one for each $p$ and for each possible fusing. In particular, for $p=1$, eq. (3.3) implies a triangle relationship for the masses of any triple involved in a fusing. It is then tempting to suppose the bound states are themselves the set of particles associated with the fields in the lagrangian and use the classical data, masses and angles, knowing already their triangle property, to set $U^c_{ab}=\theta^c_{ab}$. At least this would be sensible for those theories in which the classical mass ratios survive renormalisation, i.e., principally in the ADE series of cases. Making this identification places strong constraints on the eigenvalues $q^a_p$ for the other charges. Indeed, the only known solution when assembled into multiplets $\boldsymbol{q}_p$ to match the masses $\boldsymbol{m}=\boldsymbol{q}_1$, satisfies

$$C\boldsymbol{q}_p=4\sin^2(p\pi/2h)\,\boldsymbol{q}_p \,, \tag{3.4}$$

the eigenvalues forming the components of the other eigenvectors of the Cartan matrix when $p$ is one of the exponents of the simply laced algebra [14,20].

The fusing triangles have a succinct and attractive interpretation discovered by Dorey [20]. This has also been derived subsequently from the classical lagrangian using Lie algebraic methods [17]. In essence the idea is this. Since each particle can be associated with a spot on the Dynkin diagram, it is also reasonable to suppose that each particle can be associated with a simple root (or indeed a fundamental weight) of the algebra $g$. This by itself though is not enough to understand the coupling rule or fusing triangles. Rather, it is necessary to associate each particle with an orbit of a simple root (or its negative, see below) under the Coxeter element of the Weyl group. (For details of the Lie algebra theory used, see, for example, ref. [21].) It is useful (and always possible) to colour the Dynkin diagram of $g$ with two colours (black and white) so that each simple root of a given colour is orthogonal to every other simple root of the same colour. In fact there are just two ways do to this, and they differ merely in the interchange of black with white. A Coxeter element of $g$ is a product of Weyl reflections, one for each of the simple roots. However, up to conjugation, the order in which the reflections are performed does not matter. A particularly useful Coxeter element is defined by

$$w=\prod_{\bullet} w_{\bullet} \prod_{\circ} w_{\circ} , \tag{3.5}$$

where, in each individual product, the order is irrelevant because Weyl reflections corresponding to simple roots of a given colour commute with each other. Any Coxeter element has eigenvalues $e^{2\pi i s/h}$, where $s$ runs over the exponents of $g$.

With these definitions, a full set of $hr$ roots for $g$ is obtained as the disjoint union of the Coxeter orbits for the white simple roots, together with the Coxeter orbits of negative black simple roots. The particles are associated with these orbits, say particle $a$ is associated with the orbit $\{a\}$. Then the coupling rule [20] states simply that three particles couple classically, or fuse in the quantum theory, provided

$$\{a\}+\{b\}+\{c\}=0 . \tag{3.6}$$

This means that there is a triangle composed of roots selected one from each of the orbits $\{a\}$, $\{b\}$, $\{c\}$. Actually, if there is one there will be many such root triangles. Moreover, the set of conserved quantity conditions (3.3) are obtained from (3.6) by projection onto the eigenplanes of the Coxeter element (3.5).

A further conjecture concerns the scattering of two or more particles. If the conserved quantities survive in the quantum theory, and if each particle is distinguished from the others by at least one of the conserved quantities (not necessarily the mass), then two-particle scattering is simple in the sense that particle production is not allowed and the final particles have precisely the same momenta as the initial particles – even the possibility of momentum exchange for mass degenerate particles is ruled out [18]. Multi-particle scattering proceeds as a collection of two-particle scatterings of this simple type. Under these conditions, the initial and final two-particle states can differ by a phase (for real rapidity difference), nothing more, i.e.,

$$|\mathrm{a},\mathrm{b}\rangle_{\mathrm{out}}=S_{\mathrm{ab}}(\theta_{\mathrm{ab}})\,|\mathrm{a},\mathrm{b}\rangle_{\mathrm{in}}\,. \tag{3.7}$$

The $S$-matrix $S_{\mathrm{ab}}$ satisfies certain general requirements as a function of complex rapidity difference:

$$\text{unitarity:}\quad S_{\mathrm{a}\bar{\mathrm{b}}}^{-1}(\theta)=S_{\mathrm{ab}}(-\theta)\,,$$
$$\text{crossing:}\quad S_{\mathrm{a}\bar{\mathrm{b}}}(\theta)=S_{\mathrm{ab}}(\mathrm{i}\pi-\theta)\,. \tag{3.8}$$

These do not constrain the $S$-matrix severely, however.

A stronger constraint on the $S$-matrix is provided by insisting on a bootstrap principle similar to that leading to the constraints on the eigenvalues of the conserved charges. If two particles a, b fuse to form a third $\bar{\mathrm{c}}$, then a fourth particle scattering with the pair ab, with scattering matrix element $S_{\mathrm{da}}(\theta_{\mathrm{da}})S_{\mathrm{db}}(\theta_{\mathrm{db}})$ according to the factorisability of multi-particle scatterings mentioned above, can be regarded instead as scattering with the particle $\bar{\mathrm{c}}$ when the relative rapidity of particles a and b coincides with the fusing angle. Then the rapidities are related by

$$\theta_{\mathrm{a}}=\theta_{\mathrm{c}}-\mathrm{i}(\pi-U_{\mathrm{ac}}^{\mathrm{b}})\,,\quad \theta_{\mathrm{b}}=\theta_{\mathrm{c}}+\mathrm{i}(\pi-U_{\mathrm{bc}}^{\mathrm{a}})\,, \tag{3.9}$$

and the $S$-matrix elements are related by:

$$S_{\mathrm{d}\bar{\mathrm{c}}}(\theta_{\mathrm{dc}})=S_{\mathrm{da}}(\theta_{\mathrm{dc}}+\mathrm{i}\bar{U}_{\mathrm{ac}}^{\mathrm{b}})S_{\mathrm{db}}(\theta_{\mathrm{dc}}-\mathrm{i}\bar{U}_{\mathrm{bc}}^{\mathrm{a}})\,. \tag{3.10}$$

Equation (3.10) provides a set of consistency conditions on the $S$-matrix which has strong implications for the analytic structure of the $S$-matrix elements as functions of complex rapidity. It does not, however, imply a unique expression for the $S$-matrix.

For the simply laced Lie algebras it has been found that the classical masses and coupling data do provide solutions to the bootstrap on the understanding that direct channel bound states occur as odd-order poles in the $S$-matrix (which need not be simple poles) with a coefficient equal to a positive number times i, crossed channel bound states occurring with a coefficient of the opposite sign. Even-order poles do occur frequently, but their existence can be explained within perturbation theory as singularities of Feynman diagrams [8]. High, odd-order poles are also explained in these terms. The techniques for investigating these singularities are quite old [22], although in two dimensions, it is often convenient to work from first principles.

Although the bootstrap equations (3.10) do not have a unique solution, it is possible to conjecture a solution which is compatible with low orders of perturbation theory. For the simply laced algebras in the ADE series the conjectured solution takes the general form

$$S_{\mathrm{ab}}(\theta_{\mathrm{ab}})=S_{\mathrm{ab}}^{\mathrm{min}}(\theta_{\mathrm{ab}})\hat{S}_{\mathrm{ab}}(\theta_{\mathrm{ab}};\beta)\,, \tag{3.11}$$

in which the first factor satisfies the bootstrap but is independent of the coupling constant $\beta$ while the second factor also satisfies the bootstrap equations, depends upon the coupling constant but has no coupling dependent poles (for any real value of $\beta$) in the physical strip $\theta_{ab}=iu$, $0\leqslant u\leqslant\pi$. Moreover, $S_{ab}\to 1$ as $\beta\to 0$. In fact, every $S$-matrix element in these special cases may be written in terms of a basic "building block" $x$ defined as follows. First, define

$$(x)=\frac{\sinh(\Theta/2+i\pi x/2h)}{\sinh(\Theta/2-i\pi x/2h)}, \qquad (3.12)$$

where $\Theta$ is a generic rapidity difference. This block is certainly unitary. Next, set

$$\{x\}=\frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}, \qquad (3.13)$$

where the function $B(\beta)$ contains the coupling dependence and is given by

$$B(\beta)=\frac{1}{2\pi}\frac{\beta^2}{1+\beta^2/4\pi};$$

this is a conjecture, now verified [9] to order $\beta^4$, which is independent of the choice of Lie algebra and satisfies the requirements mentioned above for any real $\beta$.

As an example, consider the affine Toda theory based on the Lie algebra $d_4$. In this case, there are three particles of equal mass and one (corresponding to the centre dot on the Dynkin diagram) whose mass is $\sqrt{3}$ greater. Denote the light particles $\ell_1$, $\ell_2$, $\ell_3$, and the heavy one by h. Then in terms of (3.13), the $S$-matrix elements are

$$\begin{aligned} S_{\ell_a\ell_b}&=\{3\}, \quad a\neq b=1,2,3, \\ S_{\ell_a\ell_a}&=\{1\}\{5\}, \quad a=1,2,3, \\ S_{\ell_a h}&=\{2\}\{4\}, \quad a=1,2,3, \\ S_{hh}&=\{1\}\{3\}^2\{5\}. \end{aligned} \qquad (3.14)$$

The possible direct or crossed channel fusings correspond neatly to the non-zero classical couplings, $c^{\ell_1\ell_2\ell_3}$, $c^{\ell_a\ell_a h}$, $c^{hhh}$, which in turn may be traced to the coupling rule in terms of orbits of the Coxeter element, eq. (3.6). Notice too that the final pair of $S$-matrix elements have second and third order poles, respectively. These poles are explicable in terms of perturbation theory and their coefficients have been calculated [8], and shown to agree to lowest contributing order in the coupling ($\beta^4$ and $\beta^6$, respectively) with the coefficients obtained from the $S$-matrix (3.14).

## 4. Algebraic structure of the $S$-matrix

For each of the members of the ADE series of models it is possible to solve the bootstrap and obtain (at first sight lengthy) expressions for the $S$-matrix elements in terms of (3.13). However, Dorey [20] has found a unified formula that covers at least every simply laced case in terms of the roots or weights of the algebra and the Coxeter element of the Weyl group. In fact there are several expressions but, for illustration, only one will be mentioned here. Define a special set of vectors as follows:

$$\phi_a = (1 - w^{-1})\lambda_a\,, \quad a = 1, \ldots, r\,, \tag{4.1}$$

where $w$ is the Coxeter element introduced before in (3.5), and the $\lambda_a$ are the fundamental weights for the algebra. Dismantle $\{x\}$ by writing it in the form $\{x\}_+/\{x\}_-$, where the labels $\pm$ refer to the numerator and denominator of the expression for $(x)$, eq. (3.12). Then, an expression for the $S$-matrix is

$$S_{\mathrm{ab}}(\Theta) = \prod_{p=1}^{h} \{2p + 1 + \epsilon_{\mathrm{ab}}\}_+^{\lambda_{\mathrm{a}} \cdot w^{-p}\phi_{\mathrm{b}}}\,, \tag{4.2}$$

where the parameter $\epsilon_{\mathrm{ab}}$ is only sensitive to the "colours" of the particles. Specifically,

$$\epsilon_{\bullet\bullet} = \epsilon_{\circ\circ} = 0\,, \qquad \epsilon_{\circ\bullet} = +1\,, \qquad \epsilon_{\bullet\circ} = -1\,. \tag{4.3}$$

This and allied, equivalent formulae are very useful and provide new insight into the coupling formula and the structure of the bootstrap [23,24], at least in the special cases representing the ADE series. It is not difficult to verify directly that (4.2) agrees with the explicit example (3.14) given above.

The formula (4.2) is suggestive for another reason. Recall an old idea due to Zamolodchikov [18] in which a single-particle state is regarded as created from the vacuum state by the application of a creation operator corresponding to the particle in the usual way, as for free field theory. On the other hand, a two-particle state corresponding to an "in" state is given by the application of a pair of creation operators to the vacuum *in a specific order*, the opposite ordering indicating an "out" state. Then, denoting the creation operator for particle a by $A_{\mathrm{a}}(\theta_{\mathrm{a}})$, a pair of creation operators should not commute as they would for free fields but rather

$$A_{\mathrm{a}}(\theta_{\mathrm{a}})A_{\mathrm{b}}(\theta_{\mathrm{b}}) = S_{\mathrm{ab}}(\theta_{\mathrm{ab}})A_{\mathrm{b}}(\theta_{\mathrm{b}})A_{\mathrm{a}}(\theta_{\mathrm{a}})\,. \tag{4.4}$$

Note, in the present situation, the $S$-matrix is a set of numbers, one for each pair of particles, and hence assuming the associativity of the algebra of creation operators places no constraint on the $S$-matrix. This is in contrast to the situation in which the particles belong to degenerate multiplets and the scattering is non-trivial. There, the associativity implies the Yang–Baxter equations for the $S$-matrix. Since the particles are presumed to be eigenstates of the conserved quantities, the creation operators must satisfy

$$[Q_p, A_{\mathrm{a}}(\theta_{\mathrm{a}})] = q_p^{\mathrm{a}} e^{p\theta_{\mathrm{a}}} A_{\mathrm{a}}(\theta_{\mathrm{a}})\,. \tag{4.5}$$

Bearing this in mind, and recalling the possibility of fusing, suggests that for certain relative rapidities, corresponding to the fusing angles mentioned above, and to eqs. (3.9), one would perhaps expect a sort of operator product relation of the form

$$A_{\mathrm{a}}(\theta_{\mathrm{a}})A_{\mathrm{b}}(\theta_{\mathrm{b}})\approx A_{\bar{\mathrm{c}}}(\theta_{\mathrm{c}})\,. \tag{4.6}$$

Given eq. (4.6), the exchange relation (4.4) and the associativity of the particle creation operators, the bootstrap relation (3.10) follows. These facts taken together suggest it might be profitable to seek representations of the exchange relation for a given set of $S$-matrix elements, or indeed the operator product relation (4.6). Ultimately, the relationship between the fundamental fields in the lagrangian, in terms of which the perturbative field theory is defined, and the operators creating the particle states needs to be found. At present this goal seems far away.

One reason why (4.2) is appealing is its similarity to factors that may be obtained by normal-ordering products of vertex operators. To become convinced of that fact it is necessary only to consider an example. For each fundamental weight $\lambda$ consider a string-like, rapidity dependent field

$$X^{\lambda}(\theta)=\sum_{r=s+kh}\frac{h}{r}\,\mathrm{e}^{-r\theta}\lambda^{(h-s)}c_r\,, \tag{4.7}$$

where the sum extends over all integers $k$ and exponents $s$. In (4.7), the components of the weight are expressed in a basis of eigenvectors of the Coxeter element. The Fock space operators satisfy the relations

$$[c_r, c_{r'}]=(r/h)\delta_{r+r',0}\,. \tag{4.8}$$

Define a vertex operator to be the normal-ordered exponential of such a field,

$$V^{\lambda}(\theta)=\;:\exp X^{\lambda}(\theta):\;\equiv\mathrm{e}^{X^{\lambda}_{-}(\theta)}\mathrm{e}^{X^{\lambda}_{+}(\theta)}\,. \tag{4.9}$$

Then, the product of two vertex operators can be normal-ordered producing an extra factor depending upon the rapidity difference:

$$V^{\lambda}(\theta)V^{\lambda'}(\theta')$$
$$=\prod_{p=1}^{h}\,[1-\exp(\theta'-\theta+2\pi\mathrm{i}p/h)]^{\lambda\cdot w^{-p}\lambda'}:V^{\lambda}(\theta)V^{\lambda'}(\theta'):\,, \tag{4.10}$$

provided $\mathrm{Re}\,\theta>\mathrm{Re}\,\theta'$. This factor is very similar to terms in formula (4.2), particularly on recalling that $\sum_1^r w^p=0$ and noting the terms in (4.10) may be rewritten in terms of sinh functions. However, although there is a striking similarity with factors in the exchange relation, this is not quite the correct procedure since performing the normal-ordering with the vertex operators in the opposite order, making the analytic continuation to a common rapidity region, and taking the ratio of factors so obtained, gives unity rather than the $S$-matrix. Indeed, this ought to be anticipated, given the usual conformal property of the vertex operators. To obtain the $S$-matrix in a scheme such as this, it appears to be necessary to deform the vertex operator and remove the conformal property.

The conformal property may be destroyed in a number of ways but the simplest device, and one which works in the present context, is to take the annihilation part of the vertex operator (4.9) and rotate the weight appearing in the exponent of this term by applying the Coxeter element to it, obtaining a new vertex

$$V^{\lambda}(\theta) = \, :\exp X^{\lambda}(\theta): \, \equiv e^{X_-^{\lambda}(\theta)} e^{X_+^{w\lambda}(\theta)} \, . \qquad (4.11)$$

Using this vertex in the normal-ordering calculation does indeed yield factors which occur in the minimal part of (4.2) after analytic continuation and taking the ratio of the terms arising from the two orderings. Further details of this including the modifications which are necessary to accommodate the two types of colour, suggestions for incorporating the coupling constant dependence and the relationship with the conserved quantities and (4.6) may be found in ref. [25].

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